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On the motion of a small two-dimensional body submerged beneath surface waves

By P. WILMOTT
Mathematical Institute, University of Oxford, 24/29 St Giles, Oxford, OX1 3LB, UK

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The classical hydrodynamical problem of a body submerged beneath a free surface is considered. The flow is two-dimensional and the cross-section of the body and its motion are arbitrary. In the limit as a typical body dimension becomes small compared with its depth the method of matched asymptotic expansions becomes applicable and expressions for the forces and moment experienced by the body can be found. Several cases are considered in detail where the body is permitted to move in response to the forces and moment. We also find the additional forces, due to the free surface, experienced by a lifting body – a body about which there is a circulation.

1. Introduction

The history of the problem of surface gravity waves in the presence of submerged bodies begins with Lamb (1913) whose two-dimensional solution for the velocity potential for a fixed circular cylinder in a uniform stream satisfies the linearized surface conditions and assumes the cylinder to have shrunk to a point. Havelock (1927) later extended this solution, finding a further term in the limit as the circular cylinder becomes small. Havelock has also considered bodies whose surfaces may be described by the streamlines due to a distribution of sources and sinks placed in a uniform stream (Havelock 1926, 1928*a*, *b*, 1931).

The problem of the fixed circular cylinder of arbitrary size was considered by Dean (1948) and Ursell (1950). Dean showed that, in the linearized theory of small-amplitude disturbances, for a wave incident on a fixed submerged circular cylinder of arbitrary size, there is no reflected wave and the only effect at large distances from the cylinder is that the transmitted wave experiences a phase shift. The method employed was the conformal mapping of the flow region, excluding the cylinder, onto an annulus. The boundary condition at the cylinder now becomes too difficult for an analytical study and Dean turned to numerical calculation.

Ursell's method of solution involved placing systems of multipoles at the centre of the cylinder and solving for the strengths of these by two infinite sets of simultaneous equations arising from the boundary conditions on the cylinder. Uniqueness of the velocity potential is proved for any smooth boundary condition on the circular cylinder.

Ursell's method was employed by Ogilvie (1963) who permitted the cylinder to move sinusoidally in time with small amplitude. He assumed an expansion in terms of a small parameter and found the first-order oscillatory and second-order forces. Ogilvie also showed that only the first term in the outer solution is needed when calculating the time-averaged mean forces on the cylinder to second order. The oscillatory component of the second-order force does depend on the second-order potential.

Other submerged two-dimensional bodies have also been considered. For example, Evans (1970) considers the submerged vertical plate. He cites the experimental work of Keulegan & Carpenter (1958) on the formation of eddies behind bodies in oscillatory flows as the justification for assuming a potential flow.

More recently Thomas (1981) has extended the work of Ogilvie by the inclusion of a power take-off mechanism and hence considers different equations of motion for the circular cylinder's response. The mean vertical force is calculated and is found to act downwards for certain tunings of the take-off mechanism. Grue & Palm (1985) consider the radiative and diffractive effects of an oscillatory circular cylinder with both incident waves and uniform current. This work generalizes the result of Dean to find conditions for the reflection of the incident wave by a submerged circular cylinder. However, a direct comparison between this and the work to follow is not possible since the two cases involve different parameter regimes – Grue & Palm (1985) consider the uniform translation and the oscillation to be of the same order whereas here the oscillatory motion due to the incident wavetrain will be of smaller order.

In this paper we shall be concerned with finding the forces and moment experienced by a cylinder of general cross-section moving arbitrarily beneath a free surface. In particular we shall be interested in finding the body's response to such forces. In the limit as a typical body dimension becomes small compared with its depth the method of matched asymptotic expansion becomes applicable.

In §2 we state the mathematical problem which is then non-dimensionalized and scaled. The inner and outer potentials are described.

In §3 we use complex variables to describe the potential flow past a smooth two-dimensional body of arbitrary cross-section assuming that the conformal transformation which maps the body onto the unit circle is known. We then proceed to match the inner and outer potentials. The forces and moment acting on the body are found using Milne-Thomson's extension of Blasius' theorem (Milne-Thomson 1938), and specific examples considered: first the circular cylinder with forward speed, which is permitted to move in response to the forces it experiences; secondly, the motion of a body that is freely pivoted at its centre of mass; and thirdly, a neutrally buoyant body without forward speed that is free to respond to the forces and the moment.

In §4 we examine small, submerged bodies around which there is a circulation, the inner potential for which is then found. We next find the outer potential due to a vortex by introducing a delta function into the linearized momentum equation, and find the additional force on the body due to the free surface. Finally, we consider the particular example of a hydrofoil started impulsively from rest.

2. The mathematical statement of the problem

The problem is in two space dimensions only and the flow is assumed inviscid, incompressible and irrotational so that there exists a velocity potential ϕ. Let h be a typical depth of the submerged body, a a typical dimension of the body, and Ω a typical frequency of the motion. We shall take $1/\Omega$ to be the timescale of the motion of the submerged body (for example, when moving with constant velocity the speed of the body will be Ωh) except when it is without forward speed, in which case Ω will be taken to be the frequency of the incoming wavetrain ω.

The amplitude of the incident wave potential will be taken to be $\delta(h^3g)^{\frac{1}{2}}$. In this problem there are four dimensionless parameters a/h, Ω^2h/g, ω/Ω and δ. In order to make progress analytically we take the wave amplitude to be small so that $\delta \ll 1$ and also insist that $a/h \ll 1$; the dimensions of the body are therefore much less than its depth. Both $\Omega^2 h/g$ and ω/Ω will be taken to be $O(1)$.

We shall seek asymptotic solutions in terms of the parameter $\epsilon = a/h$, in two regions: the inner region, the neighbourhood of the cylinder at distances of $O(a)$, and the outer region, $O(h)$ distance away from the cylinder. Between the two regions the asymptotic matching principle must be applied.

Let the origin of axes fixed in space be at the typical depth h, with the y-axis vertically upwards. At time t the centre of mass of the body with respect to these axes is at $(\xi(t), \zeta(t))$ and the elevation of the surface above $y = h$ is $\eta(x,t)$.

The variables are made dimensionless in the following manner: $\phi = \Omega h^2 \overline{\phi}$, $r = h\overline{r}$, $\eta = h\overline{\eta}$, $t = \overline{t}/\Omega$, $\xi = h\overline{\xi}$ and $\zeta = h\overline{\zeta}$.

The usual surface conditions of conservation of mass and momentum become

$$\overline{\phi}_{\overline{y}} = \overline{\eta}_{\overline{t}} + \overline{\phi}_{\overline{x}}\overline{\eta}_{\overline{x}} \quad \text{on } \overline{y} = 1+\overline{\eta},$$

and

$$\overline{\phi}_{\overline{t}} + \tfrac{1}{2}(\overline{\phi}_{\overline{x}}^2 + \overline{\phi}_{\overline{y}}^2) + \frac{g}{\Omega^2 h}\overline{\eta} = 0 \quad \text{on } \overline{y} = 1+\overline{\eta}$$

respectively.

Coordinates moving with the body are defined by $\tilde{x} = \overline{x} - \overline{\xi}$, $\tilde{y} = \overline{y} - \overline{\zeta}$.

In considering the inner region the variables $\tilde{x}$ and $\tilde{y}$ must be rescaled by putting $\tilde{r} = \epsilon r'$ and now the dashed variables are $O(1)$ in the vicinity of the body; these are the inner variables. The outer potential will still be denoted by $\overline{\phi}$ but the inner potential is now ϕ'. The inner potential must satisfy the condition of no flow normal to the body surface.

We shall now discuss the relative orders of δ and ϵ. The disturbance at the surface due to a small body with dimension ϵh at depth h is equivalent, to lowest order, to a submerged dipole and if the Froude number $\Omega(h/g)^{\frac{1}{2}}$, is $O(1)$ then the disturbance at the surface is $O(\epsilon^2)$. However, the nonlinearity in the boundary condition is quadratic in the incident amplitude, that is, of $O(\delta^2)$. With the choice $\delta = A\epsilon$ with $A = O(1)$ these two effects are of the same order of magnitude. At the same time the motion of a neutrally buoyant body in response to incident waves is over distances of the order δh, so with our choice the motion is of the order of the body dimensions. This results, in certain situations to be discussed later, in a nonlinear problem for the body response. Of course, other choices for δ are possible and may be considered in future work.

We shall now look for asymptotic expansions of $\overline{\phi}$, ϕ' and $\overline{\eta}$ in powers of ϵ:

$$\overline{\phi} \sim \epsilon\overline{\phi}_1 + \epsilon^2\overline{\phi}_2 + \ldots, \quad \phi' \sim \epsilon\phi_1' + \epsilon^2\phi_2' + \ldots, \quad \overline{\eta} \sim \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots.$$

The first-order outer potential $\overline{\phi}_1$ is now the incident sinusoidal wavetrain potential given by $A(g/\Omega^2 h)^{\frac{1}{2}} \mathrm{e}^{-hk} \mathrm{e}^{hk\overline{y}} \cos(hk\overline{x} + (\omega/\Omega)\overline{t})$ where, to lowest order, $\omega^2 = gk$. In the above expansions we have not explicitly included $\log\epsilon$ terms which may appear. This is for two reasons: (i) we do not know *a priori* at which order they will appear; and (ii) care must be taken to asymptotically match all terms with the same algebraic order at the same time (Fraenkel 1969).

Although we are considering a two-dimensional problem we do not expect the appearance of logarithmic terms in the perturbation parameter ϵ when there is no circulation about the body, since these will only be caused by a net source of fluid, which we do not consider, and which occurs, for example, in slender-body theory. In §4 we shall examine a body with a sharp trailing edge and which will therefore have a circulation set up around it; in this case we shall allow for $\log\epsilon$ terms.

In moving coordinates the surface conditions become, on being transferred to $\tilde{y} = 1-\bar{\zeta}_0$,

$$\left(\frac{\partial}{\partial \bar{t}} - \dot{\bar{\xi}}_0(\bar{t})\frac{\partial}{\partial \tilde{x}} - \dot{\bar{\zeta}}_0(\bar{t})\frac{\partial}{\partial \tilde{y}}\right)^2 \bar{\phi}_2 + \lambda^2 \bar{\phi}_{2\tilde{y}} = 0 \quad \text{on } \tilde{y} = 1-\bar{\zeta}_0, \tag{1}$$

where $\bar{\xi} = \bar{\xi}_0 + \epsilon\bar{\xi}_1 + \ldots$, and $\bar{\zeta} = \bar{\zeta}_0 + \epsilon\bar{\zeta}_1 + \ldots$, and where $\lambda^2 = g/\Omega^2 h$ and the dot denotes differentiation with respect to $\bar{t}$. Note that the expected right-hand side quadratic in $\bar{\phi}_1$ and $\bar{\eta}_1$ is identically zero for a regular sinusoidal wave (Stoker 1957).

The problem in the outer region is

$$\nabla^2 \bar{\phi}_2 = 0 \quad \text{for } -\infty < \tilde{x} < \infty, \quad \tilde{y} < 1-\bar{\zeta}_0, (\tilde{x}, \tilde{y}) \neq (0,0)$$

subject to (1), where $\bar{\phi}_2$ is also to match with the inner potential and is to satisfy an initial condition or the radiation condition of outgoing waves only.

The inner problem is

$$\nabla^2 \phi'_i = 0$$

subject to the body boundary condition and matching with the outer potential.

Henceforth all overbars and tildes will be dropped and the outer potential will be in terms of moving coordinates. We shall retain the dashes to distinguish inner variables. The inner problem will now be considered in the complex plane, where overbars denote complex-conjugates.

3. Smooth bodies

3.1. *The inner and outer potentials*

We shall now calculate the inner and outer potentials in the case of a body about which there is no circulation. The body will be represented by a non-singular conformal mapping.

Define the complex variable $s' = x' + iy'$. It will be assumed that the unit circle can be conformally mapped onto the body by the transformations $s' = e^{i\gamma(t)}s$ and $s = f(w)$, where γ is the angle between the x' axis and a line fixed in the body. The origin of the s-plane is at the centre of mass of the body and the points at infinity of the s-plane and w-plane coincide; that is, $s \sim a_{-1}w + a_0 + a_1/w + a_2/w^2 + \ldots$ as $|w| \to \infty$ where a_{-1} is real and positive. It is also necessary that $df/dw(w) \neq 0$ for $|w| \geqslant 1$. This includes the requirement that the body has no sharp edges and that there exist potential flows past the body without circulation and with finite velocities everywhere.

Define $N(w) = f(w)\bar{f}(w^{-1})$. Expanding $N(w)$ in a Laurent expansion in powers of w we have

$$N(w) = N_1(w) + N_2(w),$$

where N_1 contains the negative powers. The complex potential for a rotation of the body in an infinite fluid at rest at infinity is then given by

$$i\dot{\gamma}(t)\, N_1(w)$$

(Milne-Thomson 1938). Therefore when the body is also translating in an infinite fluid the complex potential in scaled variables is

$$W_0(w) = \epsilon\left[\bar{I}_1 s - \left(a_{-1}\bar{I}_1 w + \frac{a_{-1}I_1}{w}\right)\right] + \epsilon^2 i\dot{\gamma}N_1(w)$$

where $I_1 = (\dot{\xi} + i\dot{\zeta})e^{-i\gamma}$.

Expanding the first-order outer potential (the incident wavetrain ϕ_1) in inner variables gives

$$\phi_1 \sim \mathrm{Re}\{\overline{H}_1+\epsilon\overline{I}_2\, s+\epsilon^2\overline{J}_3\, s^2+\epsilon^3\overline{K}_4\, s^3+\ldots\},$$

where

$$I_2 = \mathrm{i}A\lambda(hk)\exp\left\{-hk+hk\zeta+\mathrm{i}\left(hk\xi+\frac{\omega}{\Omega}t-\gamma\right)\right\},$$

$$J_3 = -\tfrac{1}{2}A\lambda(hk)^2\exp\left\{-hk+hk\zeta+\mathrm{i}\left(hk\xi+\frac{\omega}{\Omega}t-2\gamma\right)\right\},$$

and

$$K_4 = -\tfrac{1}{6}\mathrm{i}A\lambda(hk)^3\exp\left\{-hk+hk\zeta+\mathrm{i}\left(hk\xi+\frac{\omega}{\Omega}t-3\gamma\right)\right\}.$$

The term involving H_1 contains no s-dependence and will therefore give rise to a uniform pressure around the body and hence no force or moment. We shall henceforth drop all such terms. Note that in I, J and K we have not expanded ξ and ζ in powers of ϵ; this could have been done to keep the expansions consistent (and will be done in later sections as necessary), but this would only serve to further increase the length of the expression for W_0 without adding to the understanding of the source of each term in the expansion.

The inner complex potential $W(w)$ will contain both $W_0(w)$ and terms due to matching with ϕ_1 and ϕ_2, etc. To determine the inner limit of ϕ_2 we expand W_0 in outer variables to give

$$\phi_2 \sim \mathrm{Re}\left\{\frac{a_{-1}\,\mathrm{e}^{\mathrm{i}\gamma}(\overline{I}_1\, a_1-I_1\, a_{-1})}{x+\mathrm{i}y}\right\} \quad \text{as } |x+\mathrm{i}y|\to 0.$$

Again there will be a term that contains no s-dependence and that can therefore be ignored.

Therefore ϕ_2 is to match with a dipole at the origin. This may be achieved by solving

$$\nabla^2\phi_2 = a(t)\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}\delta(y)+b(t)\,\delta(x)\frac{\mathrm{d}\delta(y)}{\mathrm{d}y},$$

where δ is the Dirac delta function and

$$a+\mathrm{i}b = 2\pi a_{-1}\,\mathrm{e}^{\mathrm{i}\gamma}(\overline{I}_1\, a_1-I_1\, a_{-1}),$$

subject to the surface condition (1). The solution to this problem may be found by Fourier transform, giving

$$\phi_2 = \frac{ax+by}{2\pi(x^2+y^2)}+\frac{1}{2\pi}\int_{C_1} B(s_1,t)\exp\{\mathrm{i}s_1 x-|s_1|\, y\}\,\mathrm{d}s_1, \tag{2}$$

where $B(s_1,t)$ is a known function which can be found in the Appendix, and the inversion contour C_1 is chosen so that ϕ_2 satisfies the radiation condition.

Matching with the inner potential then proceeds by expanding the non-singular part of ϕ_2 in inner variables to give

$$\mathrm{Re}\{\epsilon\overline{I}_3\, s+\epsilon^2\overline{J}_4\, s^2+\ldots\}$$

(again the function of t alone has been dropped), where

$$\overline{I}_3 = \frac{-\mathrm{i}}{2\pi}\mathrm{e}^{\mathrm{i}\gamma}\int_{C_1} B(s_1,t)\,(s_1+|s_1|)\,\mathrm{d}s_1,$$

and

$$\overline{J}_4 = -\frac{\mathrm{e}^{2\mathrm{i}\gamma}}{4\pi}\int_{C_1} B(s,t)\, s_1(s_1+|s_1|)\,\mathrm{d}s_1.$$

We do not find ϕ_3 since its effect on the forces on the cylinder first appears at $O(\epsilon^5)$ through the ϕ_t' term in Bernoulli's equation, which is of smaller order than the terms that we shall calculate.

The matching now proceeds in a straightforward manner to give the inner complex potential to be

$$\begin{aligned} W(w) \sim\ & \epsilon\left[\bar{I}_1 s-\left(a_{-1}\bar{I}_1 w+\frac{a_{-1}I_1}{w}\right)\right]+\epsilon^2 \mathrm{i}\dot{\gamma}N_1+\epsilon^2\left[a_{-1}\bar{I}_2 w+\frac{a_{-1}I_2}{w}\right] \\ & +\epsilon^3\left[a_{-1}\bar{I}_3 w+\frac{a_{-1}I_3}{w}+a_{-1}^2\bar{J}_3 w^2+\frac{a_{-1}^2 J_3}{w^2}+2a_0 a_{-1}\bar{J}_3 w+\frac{2\bar{a}_0 a_{-1}J_3}{w}\right] \\ & +\epsilon^4\left[a_{-1}^2\bar{J}_4 w^2+\frac{a_{-1}^2 J_4}{w^2}+a_{-1}^3\bar{K}_4 w^3+\frac{a_{-1}^3 K_4}{w^3}+3a_0 a_{-1}^2\bar{K}_4 w^2+\frac{3\bar{a}_0 a_{-1}^2 K_4}{w^2}\right] \\ & +O(\epsilon^4)\text{ linear flow term}+O(\epsilon^5). \end{aligned}$$

3.2. *The forces and moment*

Since we have found the potential in the neighbourhood of the body we may apply Milne-Thomson's extension of Blasius' theorem (Milne-Thomson 1938) to find the forces and moment that the body experiences. We shall not include the hydrostatic buoyancy forces and moment in our analysis. In dimensionless, scaled, inner variables the forces are given by

$$\begin{aligned} F_x-\mathrm{i}F_y =\ & \tfrac{1}{2}\mathrm{i}\rho\Omega^2 h^3\,\mathrm{e}^{-\mathrm{i}\gamma}\,\epsilon^{-1}\int_{|w|=1}\left(\frac{\mathrm{d}W}{\mathrm{d}w}\right)^2\frac{\mathrm{d}w}{\mathrm{d}s}\mathrm{d}w \\ & +\mathrm{i}\rho\Omega^2 h^3\epsilon\frac{\partial}{\partial t}\left\{\mathrm{e}^{-\mathrm{i}\gamma}\overline{\int_{|w|=1}W\frac{\mathrm{d}s}{\mathrm{d}w}\mathrm{d}w}\right\}+\rho\Omega^2 hS(\ddot{\xi}-\mathrm{i}\ddot{\zeta}), \end{aligned}$$

where S is the cross-sectional area of the body $(O(\epsilon^2h^2))$, and an overbar denotes complex-conjugation.

The potential $W(w)$ may be substituted into this expression and the forces evaluated to $O(\epsilon^4)$ to give

$$\begin{aligned} F_x-\mathrm{i}F_y \sim\ & \epsilon^2 2\pi\rho\Omega^2 h^3\left\{(\ddot{\xi}-\mathrm{i}\ddot{\zeta})\left(\frac{S}{2\pi\epsilon^2h^2}-a_{-1}^2\right)+a_{-1}\bar{a}_1\,\mathrm{e}^{-2\mathrm{i}\gamma}(\ddot{\xi}+\mathrm{i}\ddot{\zeta}-2\mathrm{i}\dot{\gamma}(\dot{\xi}+\mathrm{i}\dot{\zeta}))\right\} \\ & +\epsilon^3 2\pi\rho\Omega^2 h^3\left\{-\mathrm{i}a_{-1}\bar{b}_1\,\mathrm{e}^{-\mathrm{i}\gamma}(\ddot{\gamma}-\mathrm{i}\dot{\gamma}^2)-2a_{-1}\,\mathrm{e}^{-\mathrm{i}\gamma}\,\bar{J}_3(a_{-1}\,\mathrm{e}^{\mathrm{i}\gamma}(\dot{\xi}+\mathrm{i}\dot{\zeta})-a_1\,\mathrm{e}^{\mathrm{i}\gamma}(\dot{\xi}-\mathrm{i}\dot{\zeta}))\right. \\ & \left.+a_{-1}\frac{\partial}{\partial t}(\mathrm{e}^{-\mathrm{i}\gamma}(a_{-1}\bar{I}_2-\bar{a}_1 I_2))\right\}+\epsilon^4 2\pi\rho\Omega^2 h^3\left\{6a_{-1}^2 a_2\bar{K}_4(\dot{\xi}-\mathrm{i}\dot{\zeta})\right. \\ & -2\mathrm{e}^{-\mathrm{i}\gamma}a_{-1}(\bar{J}_4+3a_0\bar{K}_4)\,(a_{-1}\,\mathrm{e}^{-\mathrm{i}\gamma}(\dot{\xi}+\mathrm{i}\dot{\zeta})-a_1\,\mathrm{e}^{\mathrm{i}\gamma}(\dot{\xi}-\mathrm{i}\dot{\zeta}))+2\mathrm{e}^{-\mathrm{i}\gamma}a_{-1}b_1\bar{J}_3\dot{\gamma} \\ & +a_{-1}\frac{\partial}{\partial t}(\mathrm{e}^{-\mathrm{i}\gamma}(a_{-1}\bar{I}_3-\bar{a}_1 I_3-2\bar{a}_0\bar{a}_1 J_3+2a_{-1}a_0 J_3+2a_{-1}\bar{a}_2 J_3)) \\ & \left.+2a_{-1}\,\mathrm{e}^{-\mathrm{i}\gamma}\bar{J}_3(a_{-1}I_2-a_1\bar{I}_2)\right\}+\mathrm{O}(\epsilon^5), \end{aligned} \tag{3}$$

where b_1 is the coefficient of $1/w$ in the expansion of $N_1(w)$. The terms in this expression fall into three categories: (i) those terms that do not contain I_2, I_3, J_3, J_4 or K_4 explicitly and that would therefore be present in the absence of a free surface (including added mass); (ii) terms linear in I_2, I_3, J_3, J_4 or K_4 that represent an interaction between the motion of the body and the waves (both the incident wavetrain and the waves created by the motion); and (iii) terms quadratic in I_2 and J_3 representing a nonlinear self-interaction of the incoming wavetrain.

Similarly, the moment anticlockwise is given by the real part of

$$-\tfrac{1}{2}\rho\Omega^2h^4\int_{|w|=1} s\left(\frac{\mathrm{d}W}{\mathrm{d}w}\right)^2\frac{\mathrm{d}w}{\mathrm{d}s}\mathrm{d}w+\rho\Omega^2h^4\epsilon\,\mathrm{e}^{\mathrm{i}\gamma}(\dot\xi-\mathrm{i}\dot\zeta)\times\int_{|w|=1} s\frac{\mathrm{d}W}{\mathrm{d}w}\mathrm{d}w-\rho\Omega^2h^4\epsilon^2\frac{\partial}{\partial t}\int_{|w|=1} sW\,\mathrm{d}\overline{w}, \quad (4)$$

which becomes the real part of

$$\epsilon^2 2\pi\rho\Omega^2h^4\{-\mathrm{i}a_{-1}a_1\mathrm{e}^{2\mathrm{i}\gamma}(\dot\xi-\mathrm{i}\dot\zeta)^2\}+\epsilon^3 2\pi\rho\Omega^2h^4\left\{a_{-1}b_1\mathrm{e}^{\mathrm{i}\gamma}\gamma(\dot\xi-\mathrm{i}\dot\zeta)+a_{-1}\frac{\partial}{\partial t}(\mathrm{e}^{\mathrm{i}\gamma}\overline{P}_{-1}(\dot\xi-\mathrm{i}\dot\zeta)\right.$$
$$\left.+\mathrm{e}^{-\mathrm{i}\gamma}\overline{P}_1(\dot\xi+\mathrm{i}\dot\zeta))+2\mathrm{i}a_{-1}a_1\mathrm{e}^{\mathrm{i}\gamma}(\dot\xi-\mathrm{i}\dot\zeta)\overline{I}_2)\right\}+\epsilon^4 2\pi\rho\Omega^2h^4\left\{\frac{-\mathrm{i}}{2\pi}\ddot\gamma\int_{|w|=1} sN_1\overline{\frac{\mathrm{d}s}{\mathrm{d}w}}\mathrm{d}\overline{w}\right.$$
$$-\mathrm{i}a_{-1}(\overline{I}_3+4a_0\overline{J}_3)(a_{-1}\mathrm{e}^{-\mathrm{i}\gamma}(\dot\xi+\mathrm{i}\dot\zeta)-a_1\mathrm{e}^{\mathrm{i}\gamma}(\dot\xi-\mathrm{i}\dot\zeta))$$
$$+\mathrm{i}a_{-1}\mathrm{e}^{\mathrm{i}\gamma}(\dot\xi-\mathrm{i}\dot\zeta)(a_1\overline{I}_3-a_{-1}I_3-2a_{-1}a_0J_3+2a_0a_1\overline{J}_3+2a_{-1}a_2\overline{J}_3)$$
$$\left.-a_{-1}\frac{\partial}{\partial t}(\overline{P}_{-1}\overline{I}_2+\overline{P}_1I_2)-\tfrac{1}{2}\mathrm{i}a_{-1}a_1\overline{I}_2^2\right\}+O(\epsilon^5), \quad (5)$$

where $$P_{-1}=\frac{1}{2\pi}\int_{|w|=1}\overline{s}\frac{\mathrm{d}s}{\mathrm{d}w}\frac{\mathrm{d}w}{w},\quad P_1=\frac{1}{2\pi}\int_{|w|=1}\overline{s}\frac{\mathrm{d}s}{\mathrm{d}w}w\,\mathrm{d}w.$$

The forces and moment on an arbitrarily moving two-dimensional body beneath a free surface with an incident wavetrain have been found. We shall now consider several specific examples.

3.3. *The circular cylinder*

The expression for the forces on the general body may be written in terms of hypergeometric functions as an instantaneous force plus a term that is dependent upon the history of the motion. Some simplification occurs when the body has circular cross-section. We shall first consider the cylinder to be constrained to move at constant speed and at constant depth. The assumption of potential flow cannot be justified in this case since there will generally be separation of the flow at the rear. However, the results in this section are expected to be in qualitative agreement with the equivalent results for slender submerged bodies where the assumption is more realistic.

The expressions for the forces become

$$F_x\sim\pi\rho\Omega^2h^3\left\{-\epsilon^3 2A\lambda(hk)\,\mathrm{e}^{-hk}\cos\left(hkt+\frac{\omega}{\Omega}t\right)-\epsilon^4 4\pi\lambda^3\mathrm{e}^{-2\lambda^2}+O(\epsilon^5)\right\}$$

and

$$F_y\sim\pi\rho\Omega^2h^3\left\{-\epsilon^2 2A\lambda(hk)\,\mathrm{e}^{-hk}\sin\left(hkt+\frac{\omega}{\Omega}t\right)\right.$$
$$\left.+2\epsilon^4[A^2\lambda^2\mathrm{e}^{-2hk}(hk)^3-\tfrac{1}{4}-\tfrac{1}{2}\lambda^2-\lambda^4+2\lambda^6\mathrm{e}^{-2\lambda^2}li(\mathrm{e}^{2\lambda^2})]+O(\epsilon^5)\right\}.$$

The above contains the wavemaking drag and lift found by Havelock (1928*b*), plus the forces due to the incident wavetrain. Figure 1 shows the mean upward force $\overline{F}_y$, against dimensional velocity Ωh, in the case $k=\pi/150, \epsilon=0.02, h=50, A=(3/\pi)^{\frac{1}{2}}$.

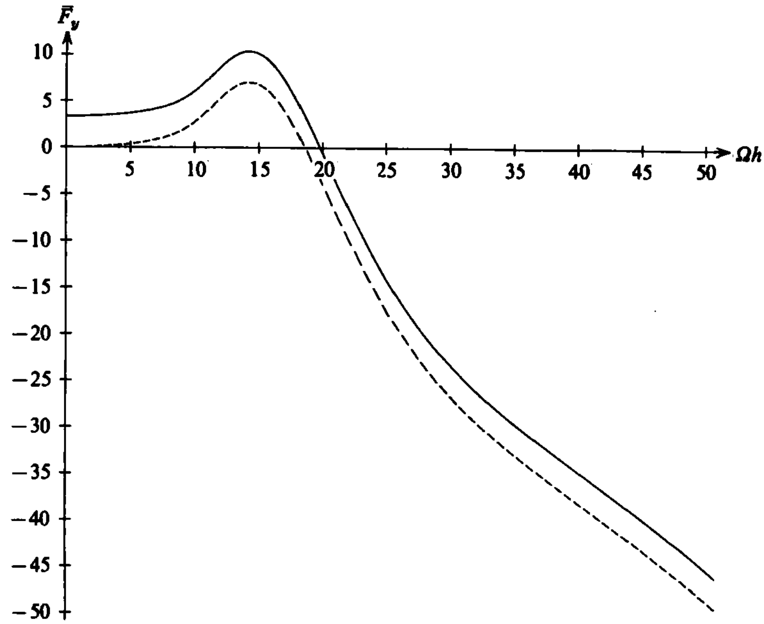

FIGURE 1. Mean upward force versus forward velocity for constrained circular cylinder: $k = \pi/150$, $\epsilon = 0.02$, $h = 50$, $\rho = 1000$ and $A = (3/\pi)^{\frac{1}{2}}$. Broken line, wavemaking force; full line, total force. There is symmetry about the vertical axis.

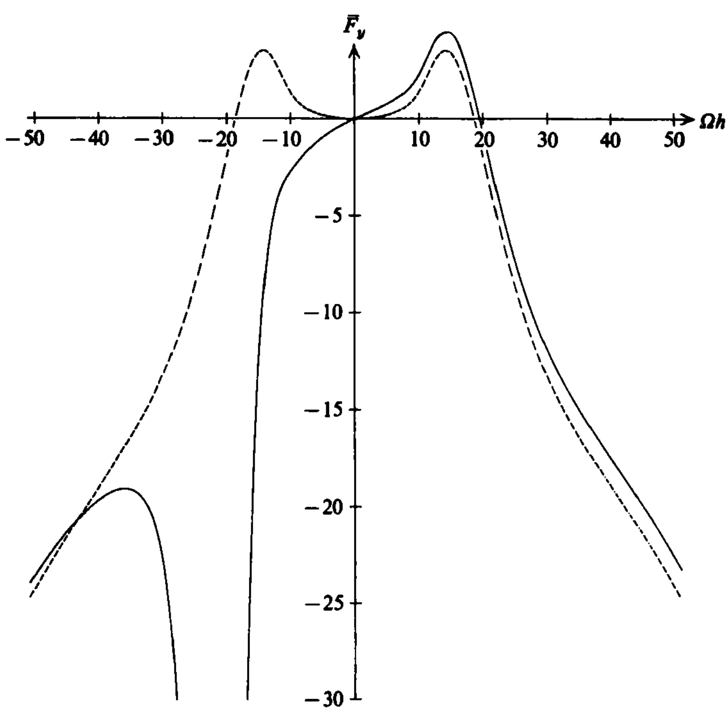

FIGURE 2. Mean upward force versus forward velocity for a neutrally buoyant circular cylinder that is free to respond: $k = \pi/150$, $\epsilon = 0.02$, $h = 50$, $\rho = 1000$ and $A = (3/\pi)^{\frac{1}{2}}$. Broken line, wavemaking force; full line, total force.

The forces experienced by a responding body have received less attention in the literature than those experienced by a restrained body. If the cylinder is neutrally buoyant and is permitted to respond to the first-order oscillatory forces so that we must solve linear ordinary differential equations for the cylinder displacement (assuming that the perturbation to the depth due to the response is small compared with the depth), the result is that

$$(\xi,\zeta) = (t,0)+\frac{\epsilon A\lambda\,\mathrm{e}^{-hk}(hk)}{\left(hk+\dfrac{\omega}{\Omega}\right)^2}\left(\frac{\omega}{\Omega}\right)\left(\cos\left(\frac{\omega}{\Omega}+hk\right)t,\sin\left(\frac{\omega}{\Omega}+hk\right)t\right),$$

and then the forces become

$$F_x \sim \pi\rho\Omega^2h^3\left\{-\epsilon^3A\lambda(hk)\,\mathrm{e}^{-hk}\frac{\omega}{\Omega}\cos\left(hkt+\frac{\omega}{\Omega}t\right)-\epsilon^4 2\pi\lambda^3\,\mathrm{e}^{-2\lambda^2}+O(\epsilon^5)\right\}$$

and

$$F_y \sim \pi\rho\Omega^2h^3\left\{-\epsilon^3A\lambda(hk)\,\mathrm{e}^{-hk}\frac{\omega}{\Omega}\sin\left(hkt+\frac{\omega}{\Omega}t\right)+\epsilon^4\left[A^2\lambda^2\,\mathrm{e}^{-2hk}(hk)^4\right.\right.$$

$$\left.\left.\times\frac{2\omega/\Omega+hk}{(hk+\omega/\Omega)^2}-\tfrac{1}{4}-\tfrac{1}{2}\lambda^2-\lambda^4+2\lambda^6\,\mathrm{e}^{-2\lambda^2}\,li(\mathrm{e}^{2\lambda^2})\right]+O(\epsilon^5)\right\}.$$

Figure 2 shows the mean contribution to the upward force against velocity, also in the case $k=\pi/150, \epsilon = 0.02, h = 50, A = (3/\pi)^{\frac{1}{2}}$. The above is only valid provided $|hk+(\omega/\Omega)| \gg \epsilon^{\frac{1}{2}}$. When $|hk+(\omega/\Omega)| = O(\epsilon^{\frac{1}{2}})$ the first-order forces are slowly varying with period $O(\epsilon^{-\frac{1}{2}})$ and the cylinder will then be perturbed about its mean depth by an amount of the same order as the depth. In this case the problem for the response of the cylinder becomes nonlinear. Such a situation represents a motion of the cylinder with the same speed as the phase speed of the incident wavetrain and in the same direction. When it occurs with surface ships it is known as 'broaching' and can lead to loss of control of the ship. Since the effect is due to the cylinder being at the same point beneath the wavetrain for long periods of time so that the first-order oscillatory forces become slowly varying, then bodies of arbitrary shape in two or three dimensions will respond in a qualitatively similar manner.

3.4. *Torque on a pivoted body*

We now return to the general case of a cylinder of arbitrary cross-section.

Experiments carried out by Keulegan & Carpenter (1958) indicate that the parameter $\pi d/2a$, where d is the distance travelled by a fluid particle during a half-cycle in the absence of the body and a is a typical body dimension, is important in determining the amount of vorticity shed by a submerged body without forward speed. In our notation $\pi d/2a = \pi A(hk)^{\frac{1}{2}}\mathrm{e}^{-hk}$. For a circular cylinder complete eddies do not form for $\pi d/2a < 15$ and for a flat plate the number is 1. This and the following section may therefore be justified provided that the body does not have a rapidly turning tangent at any point or that $\pi A(hk)^{\frac{1}{2}}\mathrm{e}^{-hk}$ is sufficiently small.

The equations of motion for the response of a neutrally buoyant cylinder are nonlinear. Much simplification occurs if the body is pivoted at its centre of mass, that is $\xi=\zeta=0$, in which case the moment is given by the real part of

$$2\pi\rho\omega^2h^4\epsilon^4\left\{\frac{-\mathrm{i}}{2\pi}\ddot{\gamma}\int_{|w|=1}sN_1\frac{\overline{\mathrm{d}s}}{\mathrm{d}w}\mathrm{d}\overline{w}-a_{-1}\frac{\partial}{\partial t}(\overline{P}_{-1}\overline{I}_2+\overline{P}_1I_2)-\tfrac{1}{2}\mathrm{i}a_{-1}a_1\overline{I}_2^2\right\}.$$

If the body is allowed to respond to this moment then the equation of motion becomes

$$R\gamma'' = 2\pi\,\mathrm{Re}\left\{-a_{-1}\frac{\partial}{\partial t}(\bar{P}_{-1}\bar{I}_2+\bar{P}_1 I_2)-\tfrac{1}{2}\mathrm{i}a_{-1}a_1\bar{I}_2^2\right\},$$

where
$$R = \rho\left\{\frac{SK^2}{\epsilon^4h^4}+\mathrm{Re}\left\{\mathrm{i}\int_{|w|=1} sN_1\frac{\overline{\mathrm{d}s}}{\mathrm{d}w}\,\mathrm{d}\bar{w}\right\}\right\},$$

K being the radius of gyration of the body with respect to the centroid. Substituting the known expression for I_2 and putting $\gamma = t+\gamma_1(t)$ gives a second-order autonomous ordinary differential equation in time for γ_1. A phase-plane analysis of this equation shows stable equilibrium points at $\gamma_1 = \alpha_0+\frac{1}{2}n\pi$, $\dot{\gamma}_1 = 0$, where $\tan 2\alpha_0 = -\arg(a_1)$ for $n = 0, 1, 2$ or 3 modulo 4 (depending upon a_{-1}, P_{-1}, P_1). The motion of the body therefore consists of a uniform rotation plus a damped oscillation.

When the body is in a stable motion with $\gamma = t+\alpha_0+\frac{1}{2}n\pi$ the inner potential is

$$W(w) \sim -\epsilon^2\mathrm{i}N_1+\epsilon^2a_{-1}\left(\bar{I}_2\,w+\frac{I_2}{w}\right)+O(\epsilon^3),$$

where $I_2 = -\mathrm{i}A\lambda\,\mathrm{e}^{-hk}(hk)\,\mathrm{e}^{-\mathrm{i}(\alpha_0+\frac{1}{2}n\pi)}$. Expanding $W(w)$ for large $|s'|$ we find that

$$\phi_3 \sim \frac{ax+by}{2\pi(x^2+y^2)} \quad \text{as } r\to 0,$$

where
$$a+\mathrm{i}b = 2\pi[-\mathrm{i}a_{-1}b_1+a_{-1}I_2-a_1\bar{I}_2]\exp\{\mathrm{i}(t+\alpha_0+\tfrac{1}{2}n\pi)\}.$$

Since I_2 is independent of time, ϕ_3 matches with a harmonically time-varying dipole of strength

$$|a_{-1}b_1+A\lambda\,\mathrm{e}^{-hk}(hk)\{a_{-1}\mathrm{e}^{-\mathrm{i}(\alpha_0+\frac{1}{2}n\pi)}+a_1\mathrm{e}^{\mathrm{i}(\alpha_0+\frac{1}{2}n\pi)}\}|.$$

It is a simple matter to show that waves produced by such a dipole travel only to the left. Therefore there are no reflected waves to $O(\epsilon^3)$ from a two-dimensional body that is pivoted at its centre of mass when in a stable periodic motion.

3.5. *Neutrally buoyant cylinder*

Next, we shall remove the restriction, imposed in the last section, that the body is pivoted at its centre of mass, and consider the response of a neutrally buoyant cylinder of general cross-section having zero forward speed. The problem for the response can be represented by three coupled ordinary differential equations for the three degrees of freedom – these equations consist of two parts: the forces and moment caused by the rotation and translation of the body, which would be present in an infinite fluid; and the forces and moment due to the incident wavetrain.

Since the amplitude of the response will be of $O(\epsilon)$ the coordinates of the centre of mass are further scaled by writing $\xi = \epsilon\xi_1$ and $\zeta = \epsilon\zeta_1$. To lowest order the expression (3) for the forces becomes

$$F_x-\mathrm{i}F_y \sim \epsilon^2 2\pi\rho\omega^2h^3\left\{(\ddot{\xi}_1-\mathrm{i}\ddot{\zeta}_1)\left(\frac{S}{2\pi\epsilon^2h^2}-a_{-1}^2\right)+a_{-1}\bar{a}_1\,\mathrm{e}^{-2\mathrm{i}\gamma}(\ddot{\xi}_1+\mathrm{i}\ddot{\zeta}_1-2\mathrm{i}\dot{\gamma}(\dot{\xi}_1+\mathrm{i}\dot{\zeta}_1))\right.$$
$$\left.-\mathrm{i}a_{-1}\bar{b}_1\,\mathrm{e}^{-\mathrm{i}\gamma}(\ddot{\gamma}-\mathrm{i}\dot{\gamma}^2)+a_{-1}\frac{\partial}{\partial t}(\mathrm{e}^{-\mathrm{i}\gamma}(a_{-1}\bar{I}_2-\bar{a}_1 I_2))\right\},$$

and the moment (5) becomes the real part of

$$\epsilon^4 2\pi\rho\omega^2 h^4 \Big\{ -\mathrm{i}a_{-1}\,a_1\,\mathrm{e}^{2\mathrm{i}\gamma}(\dot{\xi}_1-\mathrm{i}\dot{\zeta}_2)^2 + a_{-1}\,b_1\,\mathrm{e}^{\mathrm{i}\gamma}\,\dot{\gamma}(\dot{\xi}_1-\mathrm{i}\dot{\zeta}_1) + a_{-1}\frac{\partial}{\partial t}(\mathrm{e}^{\mathrm{i}\gamma}\,\overline{P}_{-1}(\dot{\xi}_1-\mathrm{i}\dot{\zeta}_1)$$
$$+\mathrm{e}^{-\mathrm{i}\gamma}\,\overline{P}_1(\dot{\xi}_1+\mathrm{i}\dot{\zeta}_1)) + 2\mathrm{i}a_{-1}\,a_1\,\mathrm{e}^{\mathrm{i}\gamma}\,\overline{I}_2(\dot{\xi}_1-\mathrm{i}\dot{\zeta}_1)$$
$$-\frac{\mathrm{i}}{2\pi}\int_{|w|=1} sN_1\,\overline{\frac{\mathrm{d}s}{\mathrm{d}w}}\,\mathrm{d}\overline{w} - a_{-1}\frac{\partial}{\partial t}(\overline{P}_{-1}\,\overline{I}_2+\overline{P}_1\,I_2) - \tfrac{1}{2}\mathrm{i}a_{-1}\,a_1\,\overline{I}_2^2\Big\}.$$

We may seek a periodic solution for the motion of the form $\xi_1+\mathrm{i}\zeta_1 = d\,\mathrm{e}^{\mathrm{i}t}, \gamma = \gamma_0+t$, where d and γ_0 are constants, representing a harmonic response with the same period as the incident wavetrain. Substitution into the above rescaled equations and equating with mass times acceleration and angular acceleration respectively yields

$$d = \delta_1 + \mathrm{e}^{\mathrm{i}\gamma_0}\frac{\overline{b}_1\,a_1-b_1\,a_{-1}}{a_1^2-a_{-1}^2}, \quad \mathrm{Re}\,\{\mathrm{i}\delta_1\,a_1\,\mathrm{e}^{2\mathrm{i}\gamma_0}+2ib_1\,\mathrm{e}^{\mathrm{i}\gamma_0}\} = 0,$$

where $\delta_1 = A\lambda(hk)\,\mathrm{e}^{-hk}$. The latter equation has two, three (with two coincident) or four real solutions for γ_0 $(0 \leqslant \gamma_0 < 2\pi)$ depending upon whether the complex number $4\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\,b_1/(\delta_1\,a_1^{\frac{1}{2}}|a_1|^{\frac{1}{2}})$ lies outside, on, or inside the curve given in the argand plane by z^3-3z^{-1}, where $|z| = 1$. For each γ_0 there is a corresponding orbit for the body. The stability of the orbits may be determined by considering a small unsteady perturbation to the steady solution. However, since the algebra involved in such an analysis is complicated we shall not address that question here. Three examples will now be given.

(i) $b_1 = 0$. For a body with $b_1 = 0$, for example an ellipse, the radius of the orbit is δ_1. The body will then follow the same path, to first order, as the fluid particle at its centre in the absence of the cylinder. Ogilvie (1963) has found this result for a small neutrally buoyant circular cylinder; here we have shown it to be true for a wider class of bodies.

We now have $\mathrm{Re}\{\mathrm{i}\delta_1\,a_1\,\mathrm{e}^{2\mathrm{i}\gamma_0}\} = 0$ and so there are four roots for γ_0, being $-0.5\arg(a_1)+\frac{1}{4}\pi+\frac{1}{2}n\pi, n = 0,1,2,3$.

(ii) $a_1 = 0$. When $a_1 = 0$ we have $d = \delta_1+b_1\,\mathrm{e}^{\mathrm{i}\gamma_0}/a_{-1}$ with $\mathrm{Re}\{2ib_1\,\mathrm{e}^{\mathrm{i}\gamma_0}\} = 0$. Therefore, there are two solutions for γ_0, $-\arg(b_1)+\frac{1}{2}\pi+n\pi, n = 0,1$ with $d = \delta_1 \pm |b_1|/a_{-1}$.

(iii) We shall give as a final example a body with both a_1 and b_1 non-zero. When $\delta_1 = 1.0$ and $a_{-1} = 1, a_1 = (1/2\surd2)(1+\mathrm{i}), a_2 = 0.1, a_n = 0, n \geqslant 3$ and a_0 is chosen so that the centre of mass is at the origin, the amplitudes of the orbits (which have been computed numerically from the equation for γ_0) are 0.900, 0.977, 1.048 and 1.089.

4. Lifting bodies

In this section we shall relax the constraint imposed in §3 that the body has no sharp edges: We shall consider those bodies with $\mathrm{d}f/\mathrm{d}w = 0$ at some point on $|w| = 1$, say $w = -1$, so that when the body is moving steadily a circulation will be set up around the body of such a magnitude that the velocity at this 'trailing edge' will be finite. A simple model (Crighton 1985) for the unsteady motion of the 'hydrofoil' is that the circulation must change with time so that the velocity at the trailing edge always remains finite – the unsteady Kutta condition. Since the circulation about the hydrofoil is changing with time vorticity will be shed creating a wake of two-dimensional vortices emanating from the trailing edge.

We shall find it necessary to assume a position for this wake that may not be its true one. In so doing we limit the accuracy to which our solution is valid. If we wish to find two terms in the inner expansion that contain expressions due to the wake then we need to know the position of the wake to $O(\epsilon)$ distance (in dimensionless variables). Unfortunately, surface waves of amplitude $O(\epsilon)$ have an $O(\epsilon)$ velocity at an $O(1)$ depth; therefore we cannot include such incoming waves. Similarly, if the circulation about the hydrofoil is $O(\epsilon)$ then distortion of the wake due to interaction with itself will again be $O(\epsilon)$. Therefore, we shall only consider the motion of a small hydrofoil at small angle of attack, which can at most be $O(\epsilon)$, beneath an otherwise calm surface.

4.1. *The inner potential for a lifting body*

First consider the motion of the hydrofoil in an infinite fluid.

Let the strength of the circulation around the aerofoil be $\Gamma(t)$ (this is called the bound vortex, as yet unknown), than at time u the aerofoil sheds a vortex of strength $-\dot{\Gamma}(u)\,\delta u$. At time t this vortex will be at $s'_{*}(t,u)$ in the s' plane and at $w_{*}(t,u)$ in the w-plane. By Milne-Thomson's (1938) circle theorem the complex potential due to such a vortex is

$$\mathrm{i}\dot{\Gamma}(u)\ln\left\{\frac{w-w_{*}(t,u)}{1-\overline{w}_{*}(t,u)\,w}\right\}\delta u+\mathrm{i}\dot{\Gamma}(u)\ln w\;\delta u;$$

therefore the potential due to the wake from time t_0 is

$$\mathrm{i}\int_{t_0-}^{t}\dot{\Gamma}(u)\ln\left\{\frac{w-w_{*}}{1-\overline{w}_{*}\,w}\right\}\mathrm{d}u+\mathrm{i}\Gamma(t)\ln w,$$

assuming $\Gamma(t_{0}-)=0$.

As in §3 the inner, non-dimensional potential for attached flow beneath a free surface is now given by the real part of

$$W\sim\epsilon\left[\overline{I}_1 s-\left(a_{-1}\overline{I}_1 w+a_{-1}\frac{I_1}{w}\right)\right]+\epsilon^2\mathrm{i}\dot{\gamma}N_1(w)+\frac{\epsilon^2\mathrm{i}}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_2(u)\ln\left\{\frac{w-w_{*}}{1-\overline{w}_{*}\,w}\right\}\mathrm{d}u$$
$$+\epsilon^2 a_{-1}\left[\overline{I}_3 w+\frac{I_3}{w}\right]+\frac{\epsilon^2\mathrm{i}}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_3(u)\ln\left\{\frac{w-w_{*}}{1-\overline{w}_{*}\,w}\right\}\mathrm{d}u+O(\epsilon^4),$$

where $I_1=(\dot{\xi}+\mathrm{i}\dot{\zeta})\,\mathrm{e}^{-\mathrm{i}\gamma}$ and I_3 comes from matching with the outer potential ϕ_2. To define Γ_2 and Γ_3 uniquely we shall invoke the Kutta condition of regular flow in the vicinity of the trailing edge. Note that the condition of $O(\epsilon)$ angle of attack becomes $I_1-\overline{I}_1=O(\epsilon)$. We shall approximate the position of the wake by the path of the trailing edge (and this approximation improves as the hydrofoil becomes thin and as the angle of attack tends to zero) and so

$$f(w_{*}(t,u))=\mathrm{e}^{\mathrm{i}\gamma(t)}\left\{c(u)+\frac{\xi(t)-\zeta(u)+\mathrm{i}\zeta(t)-\mathrm{i}\zeta(u)}{\epsilon}\right\},$$

where $c(u)=\mathrm{e}^{\mathrm{i}\gamma(u)}f(-1)$ (the ϵ appearing because in inner variables the centre of the hydrofoil is at $(1/\epsilon)\,(\xi(t),\zeta(t))$.

If we denote the inverse of the transformation f by (f^{-1}), that is, $(f^{-1})(f(w)) = w$, then the inner complex potential becomes

$$\epsilon\left[\bar{I}_1 s-\left(a_{-1}\bar{I}_1 w+a_{-1}\frac{I_1}{w}\right)\right]+\epsilon^2 \mathrm{i}\dot{\gamma}N_1(w)+\frac{\epsilon^2}{2\pi}\mathrm{i}\int_{t_0-}^{t}\{\dot{\Gamma}_2(u)+\epsilon\dot{\Gamma}_3(u)\}$$

$$\times\ln\left\{\frac{(w-(f^{-1})\,\mathrm{e}^{-\mathrm{i}\gamma(t)}\left\{\mathrm{e}^{\mathrm{i}\gamma(u)}f(-1)+\dfrac{\xi(t)-\xi(u)+\mathrm{i}\zeta(t)-\zeta(u)}{\epsilon}\right\}}{1-w(f^{-1})\,(\mathrm{e}^{\mathrm{i}\gamma(t)}\left\{\mathrm{e}^{-\mathrm{i}\gamma(u)}\bar{f}(-1)+\dfrac{\xi(t)-\xi(u)-\mathrm{i}\zeta(t)+\mathrm{i}\zeta(u)}{\epsilon}\right\}}\right\}du$$

$$+\epsilon^3 a_{-1}\left[\bar{I}_3 w+\frac{I_3}{w}\right]+O(\epsilon^4).$$

The integral may be split up into two parts, from t_0- to $t-\nu$ and from $t-\nu$ to t where $\epsilon \ll \nu \ll 1$. The integral may then be expanded asymptotically for small ϵ and hence

$$W \sim \epsilon\left[\bar{I}_1 s-\left(a_{-1}\bar{I}_1 w+a_{-1}\frac{I_1}{w}\right)\right]+\epsilon^2\mathrm{i}\dot{\gamma}N_1(w)-\frac{\epsilon^2\mathrm{i}}{2\pi}\Gamma_2(t)\ln w$$

$$+\frac{\epsilon^2\mathrm{i}}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_2(u)\ln\left[\frac{\mathrm{e}^{-2\gamma(t)}(\xi(t)-\xi(u)+\mathrm{i}\zeta(t)-\mathrm{i}\zeta(u))}{\xi(t)-\xi(u)-\mathrm{i}\zeta(t)+\mathrm{i}\zeta(u)}\right]du$$

$$-\frac{\epsilon^3 a_{-1}\mathrm{i}}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_2(u)\left\{\frac{w\,\mathrm{e}^{\mathrm{i}\gamma(t)}-(\mathrm{e}^{\mathrm{i}\gamma(u)}f(-1)-a_0)/a_{-1}}{\xi(t)-\xi(u)+\mathrm{i}\zeta(t)-\mathrm{i}\zeta(u)}\right.$$

$$\left.-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w-(\mathrm{e}^{-\mathrm{i}\gamma(u)}\bar{f}(-1)-\bar{a}_0)/a_{-1}}{\xi(t)-\xi(u)-\mathrm{i}\zeta(t)+\mathrm{i}\zeta(u)}\right\}$$

$$-\frac{\dot{\Gamma}_2(t)}{t-u}\left\{\frac{w\,\mathrm{e}^{\mathrm{i}\gamma(t)}-(\mathrm{e}^{\mathrm{i}\gamma(t)}f(-1)-a_0)/a_{-1}}{\dot{\xi}(t)+\mathrm{i}\dot{\zeta}(t)}-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w-(\mathrm{e}^{-\mathrm{i}\gamma(t)}\bar{f}(-1)-\bar{a}_0)/a_{-1}}{\dot{\xi}(t)-\mathrm{i}\dot{\zeta}(t)}\right\}du$$

$$+\frac{\mathrm{i}\epsilon^3 a_{-1}}{2\pi}\dot{\Gamma}_2(t)\ln\epsilon\left\{\frac{w\,\mathrm{e}^{\mathrm{i}\gamma(t)}-(\mathrm{e}^{\mathrm{i}\gamma(t)}f(-1)-a_0)/a_{-1}}{\dot{\xi}(t)+\mathrm{i}\dot{\zeta}(t)}\right.$$

$$\left.-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w-(\mathrm{e}^{-\mathrm{i}\gamma(t)}\bar{f}(-1)-\bar{a}_0)/a_{-1}}{\dot{\xi}(t)-\mathrm{i}\dot{\zeta}(t)}\right\}$$

$$+\mathrm{i}\epsilon^3\frac{\dot{\Gamma}_2(t)}{2\pi}\int_0^\infty\ln\left\{\frac{w-(f^{-1})\,[f(-1)+u\,\mathrm{e}^{-\mathrm{i}\gamma(t)}\,(\dot{\xi}(t)+\mathrm{i}\dot{\zeta}(t))]}{(1/w)-(f^{-1})\,[\bar{f}(-1)+u\,\mathrm{e}^{\mathrm{i}\gamma(t)}\,(\dot{\xi}(t)-\mathrm{i}\dot{\zeta}(t))]}\right\}$$

$$-\ln\left(\mathrm{e}^{-2\mathrm{i}\gamma(t)}\frac{\dot{\xi}(t)+\mathrm{i}\dot{\zeta}(t)}{\dot{\xi}(t)-\mathrm{i}\dot{\zeta}(t)}\right)+\frac{H(u-t+t_0)\,a_{-1}}{u}$$

$$\times\left\{\frac{w\,\mathrm{e}^{\mathrm{i}\gamma(t)}-(\mathrm{e}^{\mathrm{i}\gamma(t)}f(-1)-a_0)/a_{-1}}{\dot{\xi}(t)+\mathrm{i}\dot{\zeta}(t)}-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w-(\mathrm{e}^{-\mathrm{i}\gamma(t)}\bar{f}(-1)-\bar{a}_0)/a_{-1}}{\dot{\xi}(t)-\mathrm{i}\dot{\zeta}(t)}\right\}du$$

$$-\epsilon^3\mathrm{i}\frac{\Gamma_3(t)}{2\pi}\ln w+\epsilon^3 a_{-1}\left[\bar{I}_3 w+\frac{I_3}{w}\right]+O(\epsilon^4) \tag{6}$$

($H(u)$ being the unit Heaviside function), provided $t-t_0 \ll 1/\epsilon$ or $\Gamma_2 \equiv 0$ (otherwise a slight modification is needed). Because of the appearance of a $\ln\epsilon$ term in the above we introduce a 'switchback' term (Lagerstrom & Casten 1972) $-\epsilon^3/2\pi\ln\epsilon\,\mathrm{i}\Gamma_{3a}(t)\ln w$ in order to satisfy the Kutta condition.

4.2. *The outer potential due to a lifting body*

We may expand the inner potential (6) in terms of the outer variables; in so doing we find that the outer potential ϕ_2 must again match with a dipole of the same strength as in §3, as well as a vortex of strength $\Gamma_2(t)$ located at (0, 0) in outer, moving coordinates. We have already stated that matching with a dipole may be achieved by a delta function in the continuity equation. It is possible to match with an unsteady vortex and its associated trailing wake by introducing a delta function of the correct strength into the linearized momentum equation, that is

$$\left.\begin{aligned} \boldsymbol{q}_t-\dot{\xi}\boldsymbol{q}_x-\dot{\zeta}\boldsymbol{q}_y &= -\frac{\nabla p}{\rho}+\Gamma_2(t)\,\delta(x)\,\delta(y)\,(\dot{\zeta},\,-\dot{\xi}), \\ \nabla\cdot\boldsymbol{q} &= 0, \end{aligned}\right\} \tag{7}$$

where $\boldsymbol{q}$ is the velocity of the fluid, since the combination of a uniform flow and a circulation produces a force on the body normal to the flow and hence a force on the fluid in the opposite direction.

We may solve (7) subject to the boundary condition (1) by Fourier-transform methods and the solution for the potential is found to be of the form

$$\begin{aligned} &\frac{\Gamma_2}{2\pi}\tan^{-1}\frac{y}{x}-\frac{1}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_2(u)\tan^{-1}\left(\frac{y+\zeta(t)-\zeta(u)}{x+\xi(t)-\xi(u)}\right)\mathrm{d}u \\ &\quad+\frac{\Gamma_2}{2\pi}\tan^{-1}\left(\frac{y-2+2\zeta}{x}\right)-\frac{1}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_2(u)\tan^{-1}\left(\frac{y+\zeta(t)+\zeta(u)-2}{x+\xi(t)-\xi(u)}\right)\mathrm{d}u \\ &\qquad\qquad+\frac{1}{2\pi}\int_{C_2}D(s_1,t)\exp\{-\mathrm{i}s_1x-|s_1|\,y\}\,\mathrm{d}s\,\mathrm{d}u, \end{aligned}$$

where $D(s_1, t)$ may be found in the Appendix and C_2 is chosen to satisfy the radiation condition. ϕ_2 is therefore the sum of the above and (2).

In order to find I_3 we must match the inner and outer potentials to $O(\epsilon^3)$ in inner variables (details will not be included here, however). We find that

$$\begin{aligned} \bar{I}_3\,\mathrm{e}^{-\mathrm{i}\gamma} &= \frac{-a(t)+\mathrm{i}b(t)}{8\pi(1-\zeta)^2}+\frac{\Gamma_2}{2\pi(1-\zeta)}+\frac{i}{2\pi}\int_{t_0-}^{t}\frac{\dot{\Gamma}_2(u)\,\mathrm{d}u}{\xi(t)-\xi(u)+\mathrm{i}(\zeta(t)-\zeta(u)-2)} \\ &\quad+\frac{1}{2\pi}\int_{C_1}B(s_1,t)\,(s_1+|s_1|)\,\mathrm{d}s_1+\frac{1}{2\pi}\int_{C_2}D(s_1,t)\,(s_1+|s_1|)\,\mathrm{d}s_1. \end{aligned} \tag{8}$$

It now remains to find the circulations Γ_2, Γ_{3a} and Γ_3 by applying the Kutta condition. By insisting that $\mathrm{d}W/\mathrm{d}w = 0$ at $w = -1$, the trailing edge, we may equate powers of ϵ to find the values of the circulation. Once this has been done we may use Milne-Thomson's extension of Blasius' theorem to find the forces and moment on the hydrofoil as in §4. However, since the resulting expressions are rather complicated and unwieldy we shall be content with finding the additional forces on the hydrofoil that are due to the presence of the free surface. Such forces first appear via Γ_3 terms, which depend on I_3 through the Kutta condition, and are due to the downwash at the hydrofoil caused by the free surface and are therefore simply

$$F_x-\mathrm{i}F_y \sim 4\pi\mathrm{i}\epsilon^3\rho\Omega^2h^3(\dot{\xi}-\mathrm{i}\dot{\zeta})\,\mathrm{Im}\,\bar{I}_3+O(\epsilon^4),$$

this being true even for large $t-t_0$, provided the integrals in $\bar{I}_3$ exist.

4.3. *Impulsive start from rest*

We now consider a specific example when a flat-plate hydrofoil is impulsively started from rest at time $t_0 = 0$ with a fixed angle of attack $\epsilon\gamma_0$. Therefore $f(w) = w + 1/w$ and $\xi(t) = H(t), \zeta = 0$. Now $I_1 = \mathrm{e}^{-\mathrm{i}\epsilon\gamma_0}$ and so to $O(1)\, a+ib = 0$. The Kutta condition to lowest order becomes

$$\Gamma_2 = 4\pi\gamma_0 H(t).$$

Hence, from (8), for $t > 0$,

$$\bar{I}_3 = 2\gamma_0 + \frac{\mathrm{i}}{2\pi}\int_{0^-}^{t} \frac{\delta(u)\, 4\pi\gamma_0\, \mathrm{d}u}{t-u-2\mathrm{i}} + 2\gamma_0 \mathrm{i}\int_{C_1} (s_1 + |s_1|)\exp(-2|s_1| - \mathrm{i}s_1 t)\int_{0^-}^{t} \mathrm{e}^{\mathrm{i}su}\left[\cos\{\lambda|s_1|^{\frac{1}{2}}(u-t)\} - 1\right]\mathrm{d}u\,\mathrm{d}s,$$

where C_1 may be taken from $-\infty$ to $+\infty$ along the real s_1 axis.

$$\bar{I}_3 = 2\gamma_0 + \frac{2\mathrm{i}\gamma_0}{t-2\mathrm{i}} + 4\gamma_0\int_0^\infty s_1\,\mathrm{e}^{-2s_1}\left\{\frac{s_1}{s_1^2 - \lambda^2 s_1} - \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}st}\left\{\frac{\mathrm{e}^{-\mathrm{i}\lambda s_1^{\frac{1}{2}} t}}{s_1 + \lambda s_1^{\frac{1}{2}}} + \frac{\mathrm{e}^{\mathrm{i}\lambda s_1^{\frac{1}{2}} t}}{s_1 - \lambda s_1^{\frac{1}{2}}}\right\} - \frac{1}{s_1}(1 - \mathrm{e}^{-\mathrm{i}st})\right\}\mathrm{d}s_1.$$

Therefore, the additional lift due to the free surface is

$$-4\pi\epsilon^3\rho\Omega^2 h^3\left\{4\gamma_0\int_0^\infty \tfrac{1}{2}s_1\,\mathrm{e}^{-2s_1}\left\{\frac{\sin(s_1 + \lambda s_1^{\frac{1}{2}})\,t}{s_1 + \lambda s_1^{\frac{1}{2}}} + \frac{\sin(s_1 - \lambda s_1^{\frac{1}{2}})\,t}{s_1 - \lambda_1^{\frac{1}{2}}}\right\}\mathrm{d}s_1 - \frac{2\gamma_0 t}{t^2+4}\right\}.$$

In the limit at $t \to \infty$ this additional lift becomes

$$16\pi^2\epsilon^3\rho\Omega^2 h^3\gamma_0\lambda^2\,\mathrm{e}^{-2\lambda^2}.$$

5. Conclusion

The method of matched asymptotic expansions has been successfully applied to the problem of a two-dimensional body of general cross-section moving arbitrarily beneath a free surface in the limit as the ratio of a typical body dimension to its depth becomes small. When the body is smooth, so that there exist regular potential flows about it with no circulation, we have found three terms in the asymptotic expansions for the forces and moment (the lowest-order forces and moment are not affected by the free surface). We have considered several cases where the body has been permitted to respond to the forces and moment acting upon it. For example, when a uniformly translating circular cylinder responds to the forces it is found that the mean forces on the body are singular when the cylinder velocity and wave speed are the same. The waves created by a non-circular body pivoted at its centre of mass are considered and it is found that the amplitude of reflected waves is of lower order than might be expected. The restriction on the centre of mass is then removed and the amplitude of the oscillation of the body is determined. When the body has a sharp trailing edge a circulation is set up and so the outer flow will contain a point vortex. This vortex has been modelled by a delta function (point force) in the momentum equation. The additional force due to the presence of the free surface has been found and has been evaluated for an impulsive start of the body from rest.

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Appendix

The solution of the boundary-value problem

$$\nabla^2\phi_2 = a(t)\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}\delta(y)+b(t)\,\delta(x)\frac{\mathrm{d}\delta(y)}{\mathrm{d}x}$$

$$\left(\frac{\partial}{\partial t}-\dot{\xi}\frac{\partial}{\partial x}-\dot{\zeta}\frac{\partial}{\partial y}\right)^2\phi_2+\lambda^2\phi_{2_y} = 0 \quad \text{on } y = 1-\zeta,$$

(which may be found by taking the Fourier transform in x) is

$$\phi_2 = \frac{ax+by}{2\pi(x^2+y^2)}+\frac{1}{2\pi}\int_{C_1} B(s_1,t)\exp\{-\mathrm{i}s_1 x-|s_1|\,y\}\,\mathrm{d}s,$$

where

$$B = -\frac{\exp\{-2|s_1|+2|s_1|\,\zeta\}}{2|s_1|}(is_1+|s_1|\,b)-\lambda|s_1|^{-\frac{1}{2}}\exp\{-2|s_1|-\mathrm{i}s_1\xi+|s_1|\,\zeta\}\int_{t_0-}^{t}\{\mathrm{i}s_1\,a(u)$$
$$+|s_1|\,b(u)\}\exp\{\mathrm{i}s_1\,\xi(u)+|s_1|\,\zeta(u)\}\sin(\lambda|s_1|^{\frac{1}{2}}(u-t))\,\mathrm{d}u,$$

and C_1 is chosen to satisfy the radiation condition. The time at which motion begins is t_0, possibly minus infinity and $|s_1|$ is defined to be $\lim_{\nu\to 0}(s_1^2+\nu^2)^{\frac{1}{2}}$, with branch cuts along the imaginary axis such that $(s_1^2+\nu^2)^{\frac{1}{2}}$ has a positive real part in the whole cut plane. Similarly, the solution of

$$\boldsymbol{q}_t-\dot{\xi}\boldsymbol{q}_x-\dot{\zeta}\boldsymbol{q}_y = -\frac{\nabla p}{\rho}+\Gamma_2(t)\,\delta(x)\,\delta(y)\,(\dot{\zeta},\,-\dot{\xi}),$$

$$\operatorname{div}\boldsymbol{q} = 0,$$

with

$$\left(\frac{\partial}{\partial t}-\dot{\xi}\frac{\partial}{\partial x}-\dot{\zeta}\frac{\partial}{\partial y}\right)^2\phi_2+\lambda^2\phi_{2_y} = 0 \quad \text{on } y = 1-\zeta,$$

where $\boldsymbol{q} = \nabla\phi_2$ in the neighbourhood of the surface, $y = 1-\zeta$, can be found to be

$$\phi_2 = \frac{\Gamma_2}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_2(u)\tan^{-1}\left(\frac{y+\zeta(t)-\zeta(u)}{x+\xi(t)-\xi(u)}\right)\mathrm{d}u$$
$$+\frac{\Gamma_2}{2\pi}\tan^{-1}\left(\frac{y-2+2\zeta}{x}\right)\frac{-1}{2\pi}\int_{t_0-}^{t}\dot{\Gamma}_2(u)\tan^{-1}\left(\frac{y+\zeta(t)+\zeta(u)-2}{x+\xi(t)-\xi(u)}\right)\mathrm{d}u$$
$$+\frac{1}{2\pi}\int_{C_2} D(s_1,t)\exp\{-\mathrm{i}s_1 x-|s_1|\,y\}\,\mathrm{d}s,$$

where

$$D(s_1,t) = -\frac{\exp\{-s_1\xi-2|s_1|-|s_1|\,\zeta\}}{|s_1|}\int_{t_0-}^{t}\Gamma(u)\,[\cos(\lambda|s_1|^{\frac{1}{2}}(u-t))-1]$$
$$\times\{\dot{\xi}(u)\,|s_1|-\mathrm{i}\dot{\zeta}(u)\,s_1\}\exp\{\mathrm{i}s_1\,\xi(u)+|s_1\,|\zeta(u)\}\,\mathrm{d}u.$$

These two potentials may be considered to consist of multipoles along the image in the free surface of the path of the submerged body, although this interpretation is not unique.

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